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Correlation and response in a driven dissipative model

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We consider a simple dissipative system with spatial structure in contact with a heat bath. The system always exhibits correlations except in the cases of zero and maximal dissipation. We explicitly calculate the correlation function and the nonlocal response function of the system and show that they have the same spatial dependence.

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Driven dissipative systems occur in many different contexts, from collections of macroscopic particles to biological systems. Such systems are intrinsically out of equilibrium, and need the input of energy in order to remain functional. One prototype driven dissipative system is a granular gas [1], a collection of inelastic grains that dissipate energy through collisions; these have been examined extensively, experimentally [2,3], numerically [1,4–6], and analytically [7]. Because of the difficulty in treating granular gases analytically, stochastic mean-field models have been studied. One such is the Maxwell model [8], which assumes a velocity-independent collision rate and no spatial structure; this facilitates analytical calculation [9] of quantities such as the velocity distribution function and its moments.

The introduction of spatial dependence complicates and enriches the behavior, and may lead to correlations. For example, actual granular gases exhibit spatial clustering and velocity correlations because of the dissipative collisions, as is seen in simulational studies [10]. Williams and MacKintosh [11] have shown numerically that correlations exist in a one-dimensional (1D) driven dissipative gas provided the restitution coefficient is different from 1, and Soto *et al.* [12] have used the BBGKY hierarchy to study the appearance of velocity correlations in inelastic hard-sphere systems. Baldassarri *et al.* [13] and Ben-Naim and Krapivsky [14] consider a lattice variant of the Maxwell model which they solve in the freely cooling case (no driving); the latter authors calculate the spatially dependent velocity correlations which exhibit Gaussian decay with distance.

In this Rapid Communication we study a model of a driven system with spatial structure: the constituent "particles" are constrained to lie on a 1D lattice with nearest-neighbor coupling. Our main goal will be to understand the connection between the system's dissipative nature and spatial correlations. The system is coupled to a heat reservoir at temperature T, and the model is chosen so that it has a well-defined equilibrium limit for certain values of the system parameters. The mean-field version of the model, which has no spatial structure, can be solved exactly [15], in the sense that all the moments of the energy distribution may be calculated. For the model of this Rapid Communication, we calculate the two-point correlations of the system analytically, and demonstrate that nonzero correlations always exist except for the cases (a) in which there is no dissipation (in agreement with the results of Refs. [16,11]) or (b) when the dissipation is maximal. For dissipative systems driven by thermal contact with a heat bath, it is not only the bath temperature that determines the steady state of the system; the nature of the coupling to the bath is relevant as well (in contrast to nondissipative systems, for which this last plays no role in the determination of the equilibrium state). With this in mind, we calculate the response of the system to a change in one of these defining parameters, and compare the spatial dependence of the nonlocal response function to that of the correlation function.

The system we study is a generalization of a model introduced in [15], which can be regarded as the mean-field version of the case treated in this Rapid Communication. We consider N particles localized on sites n of a 1D lattice, each characterized by its energy E_n. The entire chain is coupled to an external Boltzmann-distributed bath at temperature T. The particles interact, with interactions being either between system particles or with the external bath. Specifically, a particle is chosen randomly and its interaction follows the stochastic rule

$$E_n(t+dt) = \begin{cases} \text{Value} & \text{Probability} \\ E_n(t) & 1-\Gamma dt \\ z\alpha[E_n(t)+E_{n+1}(t)] & \frac{1}{2}(1-f)\Gamma dt \\ z\alpha[E_n(t)+E_{n-1}(t)] & \frac{1}{2}(1-f)\Gamma dt \\ z[E_n(t)+E_B] & f\Gamma dt \end{cases} \quad (1)$$

Here Γ is the overall rate of interaction of a particle; it sets the time scale and is irrelevant to the steady state. f, the strength of the coupling to the bath, is a constant that determines the probability of a particle to interact with the bath, $\alpha \in [0,1]$ is a parameter characterizing the dissipation in an interaction (in analogy to a restitution coefficient), z is a stochastic variable uniformly distributed between 0 and 1, and E_B is the energy of a particle chosen randomly from the bath. In what follows, we shall only be interested in the steady states of the system.

Our main results are that (i) correlations appear in this system for all $0<f<1$ provided $\alpha<1$; when $f=0,1$ or $\alpha=1$ there are no correlations; (ii) the spatial decay of the correlation function is calculated and found to be exponential in the limit of $N\to\infty$; (iii) the static nonlocal response function (to a localized change in f or T) is proportional to the correlation function. This last point is reminiscent of the fluctuation-dissipation theorem of equilibrium statistical me-

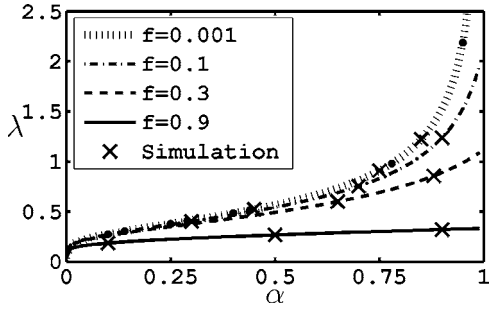

FIG. 1. The correlation length as a function of α for different values of f.

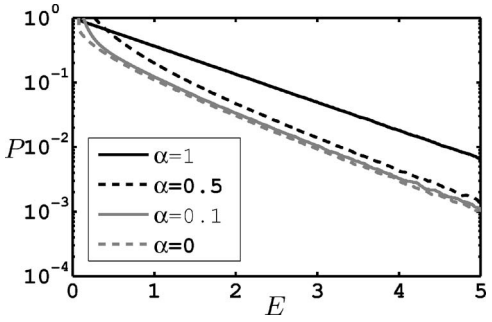

FIG. 2. The distribution function on a logarithmic scale as found from Monte Carlo simulation with $f=0.5$ and $T=1$.

chanics, but in our case there is no *a priori* reason to expect the two functions to exhibit the same spatial dependence.

Using the dynamics presented in (1) we can write equations describing the time evolution of the moments of the energy distribution and the correlation functions in the system. We will be interested in the steady-state values of the moments, so we set the time derivatives to zero. It is simple to show that the average energy is given by $\overline{E_n} \equiv \bar{E} = Tf/[2-f-2\alpha(1-f)]$, with T being the bath temperature in units where the Boltzmann constant is unity. In order to compute the correlation function $C(n,n+k) \equiv \overline{E_n E_{n+k}} - \bar{E}^2$, we need to calculate the second moments; this leads to a set of coupled equations:

$$\overline{E^2} = \frac{2\alpha^2(1-f)\overline{E_i E_{i+1}} + 2Tf\bar{E} + 2fT^2}{3 - 2\alpha^2(1-f) - f}, \tag{2}$$

$$\overline{E_n E_{n+1}} = \frac{6fT\bar{E} + (1-f)\alpha(\alpha\overline{E^2} + 3\overline{E_n E_{n+2}})}{9 - 3f - \alpha(1-f)(3+\alpha)}, \tag{3}$$

$$\overline{E_n E_{n+k}} = \frac{\frac{1}{2}\alpha(1-f)(\overline{E_n E_{n+k+1}} + \overline{E_n E_{n+k-1}}) + fT\bar{E}}{2 - \alpha(1-f) - f}. \tag{4}$$

In the last equation, $k \geq 2$. We note that if $f=0,1$ or $\alpha=1$, then for all $k>0$, $C(n,n+k)=0$: there are no correlations in these cases. This is consistent with the two-dimensional granular gas simulation of [16] where velocity correlations disappear as the restitution coefficient goes to 1, and reminiscent of similar behavior of the density correlations [11]. The case $\alpha=0$ is unique in the sense that we obtain the mean-field result for the distribution function and all correlations disappear except $C(n,n+1)$.

In what follows, we shall consider a system of N sites with cyclic boundary conditions, that is, particle $N+1$ is identified with particle 1. The results we present are valid in the limit $N \to \infty$. For general f and α, the coupled equations (2)–(4) can be written as a matrix equation of the form $\mathcal{A}\vec{W} = \vec{V}$, where $\vec{W}$ is a vector whose kth element is $\overline{E_n E_{n+k}}$, and $\vec{V}$ is a vector of constants that depend on α, f, and T. The $N \times N$ matrix $\mathcal{A}$ may be decomposed as $\mathcal{A} = \mathcal{T} - \mathcal{B}$, where $\mathcal{T}$ is a tridiagonal matrix with constant diagonals: the upper and lower diagonals are unity, while elements on the main diagonal are $2[2-a(1-f)-f]/a(f-1)$. $\mathcal{B}$ is an $N \times N$ matrix that is zero everywhere but the upper left 3×3 block (which is denoted by the 3×3 matrix $\mathcal{G}$). A tridagonal matrix having the form $\mathcal{T}$ may be analytically inverted in closed form with the help of [17]. $\mathcal{A}^{-1}$ may be computed from the relation [18]

$$\mathcal{A}^{-1} = \mathcal{T}^{-1} + \mathcal{T}^{-1}\mathcal{B}\mathcal{Q}\mathcal{T}^{-1} \tag{5}$$

where $\mathcal{Q}$ is an $N \times N$ matrix of zeros except for the upper left 3×3 block which is the matrix $(\mathcal{I} - \mathcal{R}\mathcal{G})^{-1}$, with $\mathcal{R}$ being the upper left 3×3 block of $\mathcal{T}^{-1}$, and $\mathcal{I}$ being the 3×3 unit matrix. In the limit $N \to \infty$ [19] this yields the result that the correlations decay, for all $k>1$, as

$$C(n,n+k) = D(\alpha,f)e^{-k/\lambda}. \tag{6}$$

The prefactor $D(\alpha,f)$ is a continuous function scaling the strength of the correlations, and λ is the correlation length, given by $\lambda^{-1} = \operatorname{arccosh}[(2-\alpha(1-f)-f)/\alpha(1-f)]$.

We note that the correlation length diverges as $\alpha \to 1$ and $f \to 0$, although we know that for $\alpha=1$ there are no correlations; this is because $D(\alpha,f) \to 0$ for these values. This means that around $\alpha=1$ and $f=0$ the correlations are the longest ranged but the weakest [20]. The behavior of the correlation length as a function of f and α is shown in Fig. 1, in which results from a Monte Carlo simulation of a 100-particle model are presented for comparison.

Because of the correlations, the single-particle energy distribution function $P(E)$ is not amenable to analytic calculation for general α. For $\alpha=1$, detailed balance holds, and an H theorem may be proved, and the system comes to thermal equilibrium at the bath temperature T. When $\alpha=0$, the generating function $g(\omega)$ [defined by $g(\omega) \equiv \langle e^{-\omega E} \rangle = \int_0^\infty e^{-\omega E} P(E) dE$] is the same as that of the mean-field model [15]: $g(\omega) = (\omega T + 1)\, {}_2F_1(1,2,2-f,-\omega T)$ where ${}_2F_1$ is a hypergeometric function.

For $f=0$ (and $\alpha \neq 1$) the system energy decays to zero and therefore $P(E) = \delta(0)$ is the trivial steady state. In Fig. 2, we plot the energy distribution function $P(E)$ for different values of α, as computed from a Monte Carlo simulation of 100 particles. We note that the distributions in Fig. 2 are reminiscent of those found for the mean-field case [15], that is, the distribution functions are bounded by the cases of $\alpha=0$ and 1.

For a dissipative system coupled to a bath, both the coupling strength f and the bath temperature T determine the

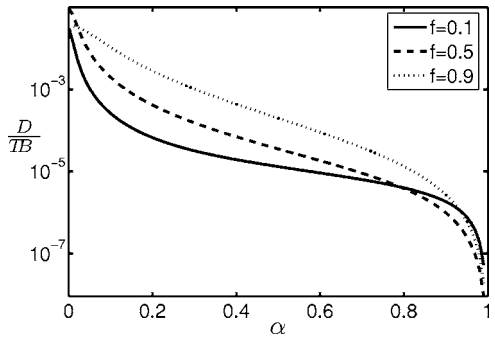

FIG. 3. Dimensionless ratio of the correlation function [Eq. (6)] and the response to a small local perturbation in bath coupling f [Eq. (7)] as a function of α, for different base values of f.

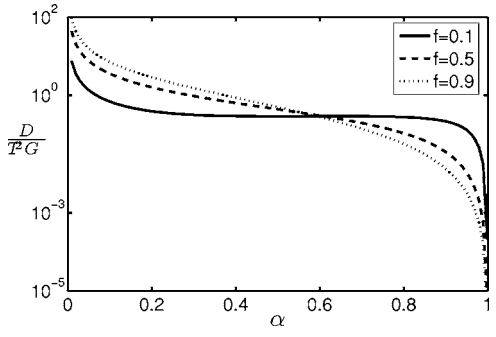

FIG. 4. Dimensionless ratio of the correlation function [Eq. (6)] and the response to a small local perturbation in bath temperature T [Eq. (8)] as a function of α, for different values of f.

steady-state behavior of the system. We will now consider the response of the system to a local change in the coupling to the bath. We stress that this is not a measurement that can be performed on a system in equilibrium, for which f plays no role in the steady state, and it is thus intrinsically a nonequilibrium measurement. Nevertheless, we shall see that, reminiscent of fluctuation-dissipation relations, the spatial response to such a perturbation is proportional to the spatial correlation function.

We imagine that each site in the system is coupled to a Boltzmann-distributed bath at temperature T with a coupling strength f. We seek to calculate the change of energy at site $n+k$ due to a small change in f (say, from f to f_0) at location n. Having changed the coupling at site n, the system no longer has translational invariance, so instead of writing the time-evolution equations for a single particle, we must consider the dynamics of the entire chain. The structure of the resulting equation [18] is again of the matrix form $\mathcal{A}'\vec{\bar{E}}=\vec{V}$, where $\vec{\bar{E}}$ is a vector of the average energies $\{\bar{E}_j\}$, and $\vec{V}$ is a vector of constants whose entries are all equal except V_n. $\mathcal{A}'$is a matrix that depends on α,f, and f_0, which has a form similar to that of $\mathcal{A}$ of Eq. (5). It may thus be inverted to yield $\bar{E}_{n+k}$, from which we obtain that for $k>1$, in the $N\to\infty$ limit,

$$\left.\frac{d\bar{E}_{n+k}}{df_0}\right|_{f_0=f} = B(\alpha,f)e^{-k/\lambda} \tag{7}$$

where $B(\alpha,f)$ is a continuous function, and where λ has the same value as in Eq. (6). This describes the nonlocal response of the system at a site a distance k from the point of a local change in coupling to the bath. We note that this intrinsically nonequilibrium response has the same spatial decay as correlation function.

We may similarly ask what the response at a site $n+k$ is to a local change in bath temperature T at site n. Using methods similar to the earlier calculations, we obtain for $k>1$, in the limit $N\to\infty$ [18],

$$\frac{d\bar{E}_{n+k}}{dT_n} = G(\alpha,f)e^{-k/\lambda} \tag{8}$$

where $G(\alpha,f)=f/[\alpha(1-f)\sinh(\lambda^{-1})]$.

In an equilibrium system, the spatiotemporal response of an observable to a small change in its conjugate field is proportional to the correlations of this variable, with the system's temperature being the constant of proportionality. Although our response measurements do not take this form, it is interesting to note that in both cases, the response of the energy $\bar{E}_{n+k}$ to a local change in the bath interaction at site n is proportional to the correlation function $C_{n,n+k}$. Moreover, for given bath temperature, the ratios of the correlation to the response (for both types of response) decreases monotonically with increasing α.

In Figs. 3 and 4 we plot dimensionless ratios of the correlation function [Eq. (6)] and the response to a local change in bath coupling f [Eq. (7)] or temperature [Eq. (8)], respectively. That is, we plot $D(\alpha,f)/TB(\alpha,f)$ and $D(\alpha,f)/T^2G(\alpha,f)$, having scaled out the dimensions with the appropriate factor of the base bath temperature. Although the qualitative form of the curves are similar, we note that $D(\alpha,f)/TB(\alpha,f)$ is significantly less than 1, while $D(\alpha,f)/T^2G(\alpha,f)$ has a significant plateau for small values of f. We note that the perturbation in the local temperature is related to the general problem of heat transport in a dissipative chain, which we will discuss separately [18].

In this Rapid Communication, we have described exact calculations for the spatial dependence of correlations and response for a model driven dissipative system. It is intriguing that, despite the fact that the fluctuation-dissipation theorem is not applicable to this system, the nonlocal response to a change in temperature or bath coupling has the same spatial dependence as the correlation function. Of course, it remains to be seen what the nature of the temporal behavior of these functions is, and it would be surprising if they were to exhibit the same frequency dependence [21].

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